A Remark on a Theorem by Kodama and Shimizu*†

A. V. Isaev

We prove a characterization theorem for the unit polydisc $\Delta^n \subset \mathbb{C}^n$ in the spirit of a recent result due to Kodama and Shimizu. We show that if M is a connected n-dimensional complex manifold such that (i) the group $Aut(M)$ of holomorphic automorphisms of M acts on M with compact isotropy subgroups, and (ii) $Aut(M)$ and $Aut(\Delta^n)$ are isomorphic as topological groups equipped with the compact-open topology, then M is holomorphically equivalent to Δ^n.

1 Introduction

For a connected complex manifold M, let $\mathrm{Aut}(M)$ denote the group of holomorphic automorphisms of M. Endowed with the compact-open topology, $\mathrm{Aut}(M)$ is a topological group. We are interested in characterizing complex manifolds by their automorphism groups.

In general, two complex manifolds M_1 and M_2 need not be holomorphically equivalent if the topological groups $\mathrm{Aut}(M_1)$ and $\mathrm{Aut}(M_2)$ are isomorphic. A simple example of this kind with non-trivial automorphism groups is given by spherical shells

$$S_r := \{z \in \mathbb{C}^n : r < ||z|| < 1\}, \quad 0 \leq r < 1.$$

It is straightforward to see that for $n \geq 2$ the group $\mathrm{Aut}(S_r)$ coincides with the unitary group U_n for all r. Next, every S_r is a Kobayashi-hyperbolic Reinhardt domain. It is shown in [Kr], [S] that two such domains are holomorphically equivalent if and only if they are equivalent by means of an elementary algebraic map, i.e. a map of the form

$$z_j \mapsto \lambda_j z_1^{a_{j1}} \cdot \ldots \cdot z_n^{a_{jn}}, \quad j = 1, \ldots, n,$$

where $\lambda_j \in \mathbb{C}^*$ and a_{jk} are integers satisfying $\det(a_{jk}) \neq 0$. An elementary algebraic map is holomorphic and one-to-one on S_r only if it is linear (i.e.

*Mathematics Subject Classification: 32M05
†Keywords and Phrases: complex manifolds, automorphism groups

reduces to dilations and a permutation of coordinates). However, S_{r_1} and S_{r_2} are not equivalent by means of such a linear map for $r_1 \neq r_2$.

If the group $\mathrm{Aut}(M)$ is sufficiently large, one can hope to obtain positive characterization results. For example, it was shown in [IK] that the space $\mathbb{C}^n$ is completely characterized by its holomorphic automorphism group as follows: if M is a connected complex manifold of dimension n and the groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\mathbb{C}^n)$ are isomorphic as topological groups, then M is holomorphically equivalent to $\mathbb{C}^n$. A similar characterization was obtained for the unit ball $B^n \subset \mathbb{C}^n$ in [I] (see also the erratum) and, under certain additional assumptions (that will be discussed below), for direct products $B^k \times \mathbb{C}^{n-k}$ in [BKS] as well as for the space $\mathbb{C}^n$ without some coordinate hyperplanes in [KS1], [KS2].

Recently, in [KS3] Kodama and Shimizu obtained the following characterization of another classical domain, the unit polydisc $\Delta^n \subset \mathbb{C}^n$ (the direct product of n copies of the unit disc $\Delta \subset \mathbb{C}$).

THEOREM 1.1 [KS3] *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. If $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Delta^n)$ are isomorphic as topological groups, then M is holomorphically equivalent to Δ^n.*

In particular, Theorem 1.1 holds for Stein manifolds and for all domains in $\mathbb{C}^n$.

The connected component of the identity $\mathrm{Aut}(\Delta^n)^0$ of the group $\mathrm{Aut}(\Delta^n)$ is isomorphic to the direct product of n copies of the group $\mathrm{Aut}(\Delta) \simeq SU_{1,1}/\mathbb{Z}_2$, and therefore contains a subgroup (which is a maximal compact subgroup) isomorphic to the n-torus $\mathbb{T}^n$. A topological group isomorphism between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\Delta^n)$ yields a smooth action by holomorphic transformations of $\mathbb{T}^n$ on M. The assumptions of holomorphic separability and smoothness of the envelope of holomorphy in Theorem 1.1 are used by the authors to linearize this action thus representing the manifold M as a Reinhardt domain in $\mathbb{C}^n$. This is possible due to a theorem by Barrett, Bedford and Dadok (see [BBD]). We note that similar assumptions were imposed on manifolds in [BKS], [KS1], [KS2] to guarantee the applicability of the result of [BBD].

It is anticipated that the assertion of Theorem 1.1 remains true if the assumptions of holomorphic separability and smoothness of the envelope of holomorphy are dropped. In this note we offer a version of Theorem 1.1 in

this direction. In particular, we do not refer to the linearization result of [BBD] in our proofs. Instead, we require that for every $p \in M$ the isotropy subgroup

$$\mathrm{Aut}_p(M) := \{g \in \mathrm{Aut}(M) : g(p) = p\}$$

is compact in $\mathrm{Aut}(M)$ and linearize the action of $\mathrm{Aut}_p(M)$ near p, which is possible due to the results of Bochner in [B] (see also [Ka]). We note that the linearizability of actions of compact groups on complex manifolds with fixed points goes back to H. Cartan (see [M] for an account of Cartan's results of this kind). In fact, we will only use the faithfulness of the isotropy representation (defined below); this statement is known as Cartan's uniqueness theorem (see [C]). The local linearizability (as opposed to the global linearizability of the $\mathbb{T}^n$-action) is sufficient to characterize Δ^n. It is not clear at this time how one could avoid using linearization arguments altogether. One difficulty here is the low-dimensionality of the maximal compact subgroup of $\mathrm{Aut}(\Delta^n)^0$. For comparison, the maximal compact subgroup of $\mathrm{Aut}(B^n)$ is isomorphic to U_n and thus has dimension n^2. This fact was of great help in [I] (see also [IK]).

Our result is the following theorem.

THEOREM 1.2 *Let M be a connected complex manifold of dimension n such that for every $p \in M$ the isotropy subgroup $Aut_p(M)$ is compact in $Aut(M)$. If $Aut(M)$ and $Aut(\Delta^n)$ are isomorphic as topological groups, then M is holomorphically equivalent to Δ^n.*

We remark that the assumption of compactness of the isotropy subgroups holds for large classes of manifolds a priori not covered by Theorem 1.1. For example, it holds whenever the action of the group $\mathrm{Aut}(M)$ on the manifold M is proper, i.e. the map

$$\mathrm{Aut}(M) \times M \to M \times M, \quad (g, p) \mapsto (g(p), p)$$

is proper. It is shown in [Ka] that $\mathrm{Aut}(M)$ acts on M properly if and only if one can find a continuous $\mathrm{Aut}(M)$-invariant distance on M. In particular, the action of $\mathrm{Aut}(M)$ is proper for all Kobayashi-hyperbolic manifolds (see also [Ko]). Hence the following holds (cf. Remark 2.1).

Corollary 1.3 *Let M be a connected Kobayashi-hyperbolic manifold of dimension n. If $Aut(M)$ and $Aut(\Delta^n)$ are isomorphic as topological groups, then M is holomorphically equivalent to Δ^n.*

2 Proof of Theorem 1.2

Let $\mathrm{Aut}(M)^0$ be the connected component of the identity of $\mathrm{Aut}(M)$. Since $\mathrm{Aut}(\Delta^n)^0$ is a Lie group of dimension $3n$ in the compact-open topology, so is $\mathrm{Aut}(M)^0$. Furthermore, every maximal compact subgroup of $\mathrm{Aut}(M)^0$ is n-dimensional and isomorphic to $\mathbb{T}^n$. For every $p \in M$ the subgroup $\mathrm{Aut}_p(M)^c := \mathrm{Aut}_p(M) \cap \mathrm{Aut}(M)^0$ is compact and therefore is contained in some maximal compact subgroup of $\mathrm{Aut}(M)^0$. Since the dimension of the $\mathrm{Aut}(M)^0$-orbit of p cannot exceed $2n$, it follows that $\dim \mathrm{Aut}_p(M)^c = n$. Hence $\mathrm{Aut}_p(M)^c$ is a maximal compact subgroup of $\mathrm{Aut}(M)^0$ (thus $\mathrm{Aut}_p(M)^c = \mathrm{Aut}_p(M)^0$), and the action of $\mathrm{Aut}(M)^0$ on M is transitive.

Let

$$\alpha_p : \mathrm{Aut}_p(M)^0 \to GL(\mathbb{R}, T_p(M)), \quad g \mapsto dg(p)$$

be the isotropy representation of $\mathrm{Aut}_p(M)^0$, where $T_p(M)$ is the tangent space to M at p and $dg(p)$ is the differential of a map g at p. Let further

$$L_p := \alpha_p \left(\mathrm{Aut}_p(M)^0\right)$$

be the corresponding linear isotropy subgroup. By the results of [C], [B], [Ka] the isotropy representation is continuous and faithful. In particular, L_p is a compact subgroup of $GL(\mathbb{R}, T_p(M))$ isomorphic to $\mathrm{Aut}_p(M)^0$. In some coordinates in $T_p(M)$ the group L_p becomes a subgroup of the unitary group U_n. Since L_p is isomorphic to $\mathbb{T}^n$, it is conjugate in U_n to the subgroup of all diagonal unitary matrices. In particular, for every $p \in M$ the group L_p contains the element $-\mathrm{id}$.

Let $\mathscr{G}$ be an $\mathrm{Aut}(M)^0$-invariant Hermitian metric on M. Since $\mathrm{Aut}(M)^0$ acts on M transitively, such a metric can be constructed by choosing an L_{p_0}-invariant positive-definite Hermitian form on $T_{p_0}(M)$ for some $p_0 \in M$, and by extending it to a Hermitian metric on all of M using the $\mathrm{Aut}(M)^0$-action (see [P] for the existence of invariant metrics for not necessarily transitive proper actions). The manifold M equipped with the metric $\mathscr{G}$ is a Hermitian symmetric space.

The theorem now follows from the general theory of Hermitian symmetric spaces (see [H]). Indeed, since the group $\mathrm{Aut}(M)^0$ acts on M with compact isotropy subgroups, contains a symmetry at every point of M, is semi-simple and is isomorphic to the direct product of n copies of the simple group $SU_{1,1}/\mathbb{Z}_2$, the manifold M is holomorphically isometric to the product of n one-dimensional irreducible Hermitian symmetric spaces (see Theorem 3.3 in

Chapter IV, Theorems 1.1 and 4.1 in Chapter V, Propositions 4.4, 5.5 and Theorem 6.1 in Chapter VIII of [H]). Clearly, each of the one-dimensional irreducible Hermitian symmetric spaces must be equivalent to the unit disc Δ, and the proof is complete. $\square$

Remark 2.1 One can obtain Corollary 1.3 without referring to the theory of Hermitian symmetric spaces. Indeed, as in the proof of Theorem 1.2, we see that M is homogeneous. Hence, by the (non-trivial) result of [N], the manifold M is holomorphically equivalent to a bounded domain in $\mathbb{C}^n$. Corollary 1.3 now follows from Theorem 1.1.

References

[BBD] Barrett, D. E., Bedford, E. and Dadok, J., T^n-actions on holomorphically separable complex manifolds, *Math. Z.* 202(1989), 65–82.

[BKS] Byun, J., Kodama, A. and Shimizu, S., A group-theoretic characterization of the direct product of a ball and a Euclidean space, *Forum Math.* 18(2006), 983–1009.

[B] Bochner, S., Compact groups of differentiable transformations, *Ann. of Math.* 46(1945), 372–381.

[C] Cartan, H., Les fonctions de deux variables complexes et le problème de la représentation analytique, *J. Math. Pures Appl. (9)* 10(1931), 1–114.

[H] Helgason, S., *Differential Geometry and Symmetric Spaces*, Academic Press, 1962.

[I] Isaev, A. V., Characterization of the unit ball in $\mathbb{C}^n$ among complex manifolds of dimension n, *J. Geom. Analysis* 14(2004), 697–700.

[IK] Isaev, A. V. and Kruzhilin, N. G., Effective actions of the unitary group on complex manifolds, *Canad. J. Math.* 54 (2002), 1254–1279.

[Ka] Kaup, W., Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen, *Invent. Math.* 3(1967), 43–70.

[Ko] Kobayashi, S., *Hyperbolic Manifolds and Holomorphic Mappings*, Marcel Dekker, New York, 1970.

[KS1] Kodama, A. and Shimizu, S., A group-theoretic characterization of the space obtained by omitting the coordinate hyperplanes from the complex Euclidean space, *Osaka J. Math.* 41(2004), 85–95.

[KS2] Kodama, A. and Shimizu, S., A group-theoretic characterization of the space obtained by omitting the coordinate hyperplanes from the complex Euclidean space, II, *J. Math. Soc. Japan* 58(2006), 643–663.

[KS3] Kodama, A. and Shimizu, S., An intrinsic characterization of the unit polydisc, to appear in *Michigan Math. J.*

[Kr] Kruzhilin, N. G., Holomorphic automorphisms of hyperbolic Reinhardt domains (translated from Russian), *Math. USSR-Izv.* 32(1989), 15–38.

[M] Martin, W. T., Mappings by means of systems of analytic functions of several complex variables, *Bull. Amer. Math. Soc.* 50(1944), 5–19.

[N] Nakajima, K., Homogeneous hyperbolic manifolds and homogeneous Siegel domains, *J. Math. Kyoto Univ.* 25(1985), 269–291.

[P] Palais, R. S., On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.* 73(1961), 295–323.

[S] Shimizu, S., Automorphisms of bounded Reinhardt domains, *Japan. J. Math.* 15(1989), 385–414.

Department of Mathematics
The Australian National University
Canberra, ACT 0200
AUSTRALIA
E-mail: alexander.isaev@maths.anu.edu.au